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An analytical solution for consolidation with vertical drains under multi-ramp loading

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Various analytical solutions have been proposed for a unit-cell consolidation with a vertical drain under surcharge loading. These solutions involve different assumptions to address various aspects of consolidation. There is a lack of generalised solution for analysing consolidation of soil assisted by a vertical drain under various loading conditions. This paper presents a simplified solution for consolidation under multi-ramp loading. Generalised governing equations of equal-strain consolidation are solved. Simultaneous radial and vertical flow conditions, as well as the combined effects of drain resistance and smear, are taken into account fully. An increase in total stress due to multi-ramp loading is reasonably modelled as a function of both time and depth. An analytical solution to calculate excess pore-water pressure at any arbitrary point in soil is derived by using the method of separation of variables. The conventional definition of the degree of consolidation is given in terms of the dissipation of excess pore-water pressure as a result of the maximum increase in total stress in soil. This definition is interpreted in relation to the ultimate ground surface settlement due to surcharge preloading. Its validity and accuracy are verified by comparing the special cases of the proposed solution with two available analytical solutions. The proposed solution is also validated by a well-documented case history with settlement and pore-water pressure measurements. Reasonably good agreement is obtained. A new degree of dissipation is defined in terms of the dissipation of excess pore-water pressure as a result of currently induced increase in total stress in soil. By using this definition, an equation is proposed to estimate the gain in undrained strength of soil due to consolidation for assessing the stability of surcharge fills more effectively and correctly. The loading path over time and the compressibility of smeared soil are shown to have a potentially important influence on the degree of consolidation and the degree of dissipation.

KEYWORDS: compressibility; consolidation; embankments; ground improvement; permeability; pore pressures

INTRODUCTION

Surcharge preloading is a traditional ground treatment technique widely used to enhance the shear strength and reduce the compressibility of soft, fine-grained soils due to consolidation and the associated increase in effective stress. The post-construction foundation settlement can be significantly eliminated not only in the primary consolidation but also in the secondary consolidation of the soil (Almeida *et al.*, 2000; Alonso *et al.*, 2000). Vertical prefabricated drains (or nowadays less popular sand/stone columns) are commonly utilised to accelerate the consolidation in surcharge preloading.

Analytical solutions predicting the extent of consolidation in surcharge preloading play an important role in preliminary design of vertical drains and surcharge fills. Since the pioneering work of Barron (1948), finding a way to predict the extent of a unit-cell consolidation with a vertical drain has captured the attention of the ground improvement community. Many solutions have been proposed based on various assumptions and considerations. Among them, a large number of solutions were derived for the consolidation of soil subjected to a uniform increase in total stress under instantaneous step load(s) (Barron, 1948; Yoshikuni & Nakanodo, 1974; Hansbo, 1981, 2001; Onoue, 1988; Zeng & Xie, 1989; Xie *et al.*, 1994; Chai *et al.*, 1997; Wang & Jiao, 2004; Indraratna *et al.*, 2005a, 2005b, 2005d, 2008; Walker & Indraratna, 2006, 2007; Rujikiatkamjorn & Indraratna, 2007; Walker *et al.*, 2012; Deng *et al.*, 2013a, 2013b; Kianfar *et al.*, 2013; Tang *et al.*, 2013). However, in practical situations, surcharge loading is almost always gradually and incrementally applied. The total stress in soil increases synchronously with the increase in surcharge loading. Such loading conditions would be more appropriately modelled as a time-dependent increase in total stress under multi-ramp loading. Theoretically, strictly speaking, a uniformly distributed increase in total stress with depth should be used under radial strain-restricted one-dimensional compression conditions of the unit-cell consolidation theory. In practice, however, the increase in total stress decreases progressively with the increase in depth, especially when the width of surcharge fills is narrow relative to the thickness of soil. In this regard, the loading conditions may be modelled as not only time-dependent but also depth-varying increase in total stress.

For the consolidation under ramp loading, only a limited number of analytical solutions have been proposed. Table 1 shows the available solutions for the simplest case of consolidation assuming the mechanical and hydraulic properties of soil and drain are uniform and constant. All solutions were derived from simplified governing equations, as shown in the second column of Table 1. These equations were established

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Table 1. Available analytical solutions of the consolidation with vertical drains under ramp loading assuming constant material properties

References	Governing equations	Assumptions and remarks	Drain resistance	Smear	Stress distribution	Loading
Schiffman (1959); modified by Kurma Rao & Vijaya Rama Raju (1990)	$\begin{cases} \dfrac{k_v}{\gamma_w}\dfrac{\partial^2 u}{\partial z^2} = -m_v\left[R - \dfrac{\partial u}{\partial t}\right] \\ \dfrac{k_h}{\gamma_w}\left(\dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r}\right) = -m_v\left[R - \dfrac{\partial \bar{u}}{\partial t}\right] \\ \dfrac{k_{sh}}{\gamma_w}\left(\dfrac{\partial^2 u_s}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u_s}{\partial r}\right) = -m_{sv}\left[R - \dfrac{\partial \bar{u}_s}{\partial t}\right] \end{cases}$	• Equal strain. • R is the constant loading rate. • Only the solution at the end of construction is presented. • For combined vertical and radial flow, the Carrillo (1942) method is applied. • The solution is developed only for the design of vertical drain spacing to achieve a target degree of consolidation within a given construction time period.	Ignored	Considered, but vertical flow is ignored	Depth-invariant	Single-ramp
Olson (1977)	$\begin{cases} \dfrac{k_v}{\gamma_w}\dfrac{\partial^2 u}{\partial z^2} = m_v\dfrac{\partial u}{\partial t} \\ \dfrac{k_h}{\gamma_w}\left(\dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r}\right) = m_v\dfrac{\partial \bar{u}}{\partial t} \end{cases}$	• Equal strain. • For single-ramp loading, solutions are approximately derived by separately integrating the differential pore pressure–load–time relationships from the Terzaghi & Fröhlich (1936) and the Barron (1948) solutions. • For combined vertical and radial flow, the Carrillo (1942) method is applied. • For multi-ramp loading, the superposition method is applied.	Ignored	Ignored	Depth-invariant	Multi-ramp
Zhu & Yin (2001a, 2001b)	$\dfrac{k_h}{\gamma_w}\left(\dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r}\right) + \dfrac{k_v}{\gamma_w}\dfrac{\partial^2 u}{\partial z^2} = -m_v\left[\dfrac{d\sigma(t)}{dt} - \dfrac{\partial u}{\partial t}\right]$	• Free strain.	Ignored	Ignored	Depth-invariant	Single-ramp
Zhu & Yin (2004)	$\begin{cases} \dfrac{k_h}{\gamma_w}\left(\dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r}\right) + \dfrac{k_v}{\gamma_w}\dfrac{\partial^2 u}{\partial z^2} = -m_v\left[\dfrac{d\sigma(t)}{dt} - \dfrac{\partial u}{\partial t}\right] \\ \dfrac{k_{sh}}{\gamma_w}\left(\dfrac{\partial^2 u_s}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u_s}{\partial r}\right) + \dfrac{k_v}{\gamma_w}\dfrac{\partial^2 u_s}{\partial z^2} = -m_v\left[\dfrac{d\sigma(t)}{dt} - \dfrac{\partial u_s}{\partial t}\right] \end{cases}$	• Free strain.	Ignored	Considered, but $m_{sv} = m_v$, $k_{sv} = k_v$	Depth-invariant	Single-ramp

Lei & Jiang (2005), extended from Leo (2004)	$\begin{cases} \dfrac{k_\mathrm{h}}{\gamma_\mathrm{w}}\left(\dfrac{\partial^2 u}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial u}{\partial r}\right)+\dfrac{k_\mathrm{v}}{\gamma_\mathrm{w}}\dfrac{\partial^2 u}{\partial z^2}=-m_\mathrm{v}\left[\dfrac{\partial\sigma(z,t)}{\partial t}-\dfrac{\partial\bar{u}}{\partial t}\right] \\ \dfrac{k_\mathrm{sh}}{\gamma_\mathrm{w}}\left(\dfrac{\partial^2 u_\mathrm{s}}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial u_\mathrm{s}}{\partial r}\right)=0 \end{cases}$	• Equal strain. • Leo (2004) assumes uniform stress, but Lei & Jiang (2005) consider depth-varying stress.	Considered	Considered, but vertical flow and compression are ignored	Depth-varying	Single-ramp
Conte & Troncone (2009)	$\begin{cases} \dfrac{k_\mathrm{v}}{\gamma_\mathrm{w}}\dfrac{\partial^2 u}{\partial z^2}=-m_\mathrm{v}\left[\dfrac{\mathrm{d}\sigma(t)}{\mathrm{d}t}-\dfrac{\partial u}{\partial t}\right] \\ \dfrac{k_\mathrm{h}}{\gamma_\mathrm{w}}\left(\dfrac{\partial^2 u}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial u}{\partial r}\right)=-m_\mathrm{v}\left[\dfrac{\mathrm{d}\sigma(t)}{\mathrm{d}t}-\dfrac{\partial\bar{u}}{\partial t}\right] \\ \dfrac{k_\mathrm{sh}}{\gamma_\mathrm{w}}\left(\dfrac{\partial^2 u_\mathrm{s}}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial u_\mathrm{s}}{\partial r}\right)=0 \end{cases}$	• Equal strain. • Solution of the first governing equation was obtained by Conte & Troncone (2006). • For combined vertical and radial flow, the Carrillo (1942) method is applied. • A general time-dependent loading is assumed by using the Fourier series.	Considered	Considered, but vertical flow and compression are ignored	Depth-invariant	Multi-ramp
Walker & Indraratna (2009)	$\left[\mathrm{d}T_\mathrm{h}\dfrac{\eta}{\bar{\eta}}\bar{u}-\mathrm{d}T_\mathrm{v}\dfrac{\partial}{\partial Z}\left(\dfrac{k_\mathrm{v}}{\bar{k}_\mathrm{v}}\dfrac{\partial\bar{u}}{\partial Z}\right)\right]=\dfrac{m_\mathrm{v}}{\bar{m}_\mathrm{v}}\dfrac{\partial(\bar{\sigma}-\bar{u}_\mathrm{r})}{\partial t}$ For details of the notation, see the source reference.	• Equal strain. • Vertical flow is assumed to be governed by the average vertical hydraulic gradient. • The governing equation is derived by incorporating vertical flow into the approximate approach of Hansbo (1981).	Ignored	Considered, but $m_\mathrm{sv}=m_\mathrm{v}$, $k_\mathrm{sv}=k_\mathrm{v}$	Depth-varying	Multi-ramp
Lu *et al.* (2011), extended from Tang & Onitsuka (2000)	$\begin{cases} \dfrac{k_\mathrm{h}}{\gamma_\mathrm{w}}\left(\dfrac{\partial^2 u}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial u}{\partial r}\right)+\dfrac{k_\mathrm{v}}{\gamma_\mathrm{w}}\dfrac{\partial^2 \bar{u}_\mathrm{r}}{\partial z^2}=-m_\mathrm{v}\left[\dfrac{\mathrm{d}\sigma(z,t)}{\mathrm{d}t}-\dfrac{\partial\bar{u}_\mathrm{r}}{\partial t}\right] \\ \dfrac{k_\mathrm{sh}}{\gamma_\mathrm{w}}\left(\dfrac{\partial^2 u_\mathrm{s}}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial u_\mathrm{s}}{\partial r}\right)+\dfrac{k_\mathrm{v}}{\gamma_\mathrm{w}}\dfrac{\partial^2 \bar{u}_\mathrm{r}}{\partial z^2}=-m_\mathrm{v}\left[\dfrac{\mathrm{d}\sigma(z,t)}{\mathrm{d}t}-\dfrac{\partial\bar{u}_\mathrm{r}}{\partial t}\right] \end{cases}$ $\bar{u}_r$ is the average excess pore-water pressure between r_d and r_e at a given depth.	• Equal strain. • Vertical flow is assumed to be governed by the average vertical hydraulic gradient. • Tang & Onitsuka (2000) assume uniform stress, but Lu *et al.* (2011) consider depth-varying stress.	Considered	Considered, but $m_\mathrm{sv}=m_\mathrm{v}$, $k_\mathrm{sv}=k_\mathrm{v}$	Depth-varying	Multi-ramp
Indraratna *et al.* (2011), extended from Lekha *et al.* (1998)	$\begin{cases} \dfrac{k_\mathrm{h}}{\gamma_\mathrm{w}}\left(\dfrac{\partial^2 u}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial u}{\partial r}\right)=-m_\mathrm{v}\left[\dfrac{\mathrm{d}\sigma(t)}{\mathrm{d}t}-\dfrac{\partial\bar{u}}{\partial t}\right] \\ \dfrac{k_\mathrm{sh}}{\gamma_\mathrm{w}}\left(\dfrac{\partial^2 u_\mathrm{s}}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial u_\mathrm{s}}{\partial r}\right)=-m_\mathrm{v}\left[\dfrac{\mathrm{d}\sigma(t)}{\mathrm{d}t}-\dfrac{\partial\bar{u}_\mathrm{s}}{\partial t}\right] \end{cases}$	• Equal strain. • Vertical flow is ignored. • Lekha *et al.* (1998) ignore the smear effect, Indraratna *et al.* (2011) consider it.	Ignored	Considered, but $m_\mathrm{sv}=m_\mathrm{v}$, vertical flow is ignored	Depth-invariant	Single-ramp

based on various assumptions on flow conditions, stress distributions and loading conditions, as shown in the third, sixth and seventh columns of Table 1, respectively. Various considerations were also given to the effects of drain resistance and smear, as shown in the fourth and fifth columns of Table 1, respectively. For example, a load continuously increasing at a constant rate was assumed by Schiffman (1959) and Kurma Rao & Vijaya Rama Raju (1990). The Carrillo (1942) approach was applied by Olson (1977) to approximately combine radial and vertical flows. Vertical flow was ignored by Lekha *et al.* (1998) and Indraratna *et al.* (2011). Drain resistance was ignored in most of the solutions. Smear effect was generally considered. However, the volume compressibility and vertical hydraulic conductivity of smeared soil were either ignored (Leo, 2004; Lei & Jiang, 2005; Conte & Troncone, 2009) or assumed to be the same as those of undisturbed soil (Zhu & Yin, 2004; Walker & Indraratna, 2009; Lu *et al.*, 2011). Strictly speaking, only Walker & Indraratna (2009) and Lu *et al.* (2011) considered a depth-varying increase in total stress in soil under multi-ramp loading. Nevertheless, their solutions were derived from governing equations in different forms from others. Vertical flow was assumed to be governed by the average vertical hydraulic gradient. It is evident from Table 1 that the available solutions were obtained based on various simplifying approximations to governing equations.

There are only four analytical solutions available specifically for consolidation under multi-ramp loading in the literature, namely Olson (1977), Conte & Troncone (2009), Walker & Indraratna (2009) and Lu *et al.* (2011). However, drain resistance and smear effect were ignored by Olson (1977). Vertical flow and compression of smeared soil were not considered by Conte & Troncone (2009). Drain resistance was ignored by Walker & Indraratna (2009), whereas vertical flow was assumed to be governed by an average vertical hydraulic gradient by Walker & Indraratna (2009) and Lu *et al.* (2011). On the basis of this assumption, only an average excess pore-water pressure was obtained at a given depth. The excess pore-water pressure at any specific point in soil cannot be obtained.

In this paper, a simplified analytical solution is derived from the generalised governing equations of equal-strain consolidation assuming uniform and constant material properties. The consolidation of soil is subjected to an increase in total stress with depth under multi-ramp loading. Combined effects of drain resistance and smear are fully taken into account. The excess pore-water pressure at any arbitrary point in soil is obtained. The validity and accuracy of the proposed solution are verified by comparing the special cases of the proposed solution with two available analytical solutions – the free-strain solution derived by Zhu & Yin (2004) and the equal-strain solution derived by Tang & Onitsuka (2000). The proposed solution is also validated by comparing calculated results with some reported field data for a test fill embankment at the Chek Lap Kok international airport in Hong Kong. The degree of consolidation is defined in terms of the dissipation of excess pore-water pressure as a result of the maximum increase in total stress. A new degree of dissipation is defined in terms of the dissipation of excess pore-water pressure as a result of currently induced increase in total stress. The former definition is interpreted in relation to the ultimate ground surface settlement due to surcharge preloading. The latter is newly interpreted in relation to the gain in shear strength of soil, and consequently to the stability of surcharge fills. The effect of multi-ramp loading on the degree of consolidation and the degree of dissipation is investigated, together with the effect of the volume compressibility of smeared soil.

PROBLEM DESCRIPTION

Figure 1 shows a unit-cell model for the consolidation with a vertical drain. The soil is subjected to a depth-varying and time-dependent increase in total stress under multi-ramp loading. The governing equations of equal-strain consolidation assuming constant material properties (Terzaghi, 1943) are given in full by

$$\frac{k_\mathrm{h}}{\gamma_\mathrm{w}}\left[\frac{\partial^2 u(r,z,t)}{\partial r^2}+\frac{1}{r}\frac{\partial u(r,z,t)}{\partial r}\right]+\frac{k_\mathrm{v}}{\gamma_\mathrm{w}}\frac{\partial^2 u(r,z,t)}{\partial z^2} = -m_\mathrm{v}\left[\frac{\partial \sigma(z,t)}{\partial t}-\frac{\partial \bar{u}(z,t)}{\partial t}\right], \quad r_\mathrm{s}\le r\le r_\mathrm{e} \tag{1}$$

$$\frac{k_\mathrm{sh}}{\gamma_\mathrm{w}}\left[\frac{\partial^2 u_\mathrm{s}(r,z,t)}{\partial r^2}+\frac{1}{r}\frac{\partial u_\mathrm{s}(r,z,t)}{\partial r}\right]+\frac{k_\mathrm{sv}}{\gamma_\mathrm{w}}\frac{\partial^2 u_\mathrm{s}(r,z,t)}{\partial z^2} = -m_\mathrm{sv}\left[\frac{\partial \sigma(z,t)}{\partial t}-\frac{\partial \bar{u}_\mathrm{s}(z,t)}{\partial t}\right], \quad r_\mathrm{d}\le r\le r_\mathrm{s} \tag{2}$$

where r and z are the radial and vertical coordinates, respectively; t is the time; r_d, r_s and r_e are the radii of the vertical drain, the smear zone and the effective influence zone of the vertical drain, respectively; u and u_s are the excess pore-water pressure of undisturbed soil and smeared soil, respectively; σ is the increase in total stress in soil due to surcharge loading and unloading; $\bar{u}$ and $\bar{u}_\mathrm{s}$ are the average excess pore-water pressures at a given depth in the radial direction between r_s and r_e and between r_d and r_s, respectively; k_h, k_v and m_v are the horizontal and vertical hydraulic conductivity and volume compressibility of the undisturbed soil, respectively; k_sh, k_sv and m_sv are the horizontal and vertical hydraulic conductivity and volume compressibility of the smeared soil, respectively; and γ_w is the unit weight of water.

It is worth noting that equal strain (assumption) gives an average vertical strain term to the governing equations of consolidation, but it is not vice versa. In other words, the governing equations established based on the equal-strain assumption do not guarantee that the equal-strain condition during the consolidation process can be maintained by their solution. This can be readily proved by the calculated average vertical strain along the radial direction, which is by no means uniform. Nevertheless, it has been well recognised that

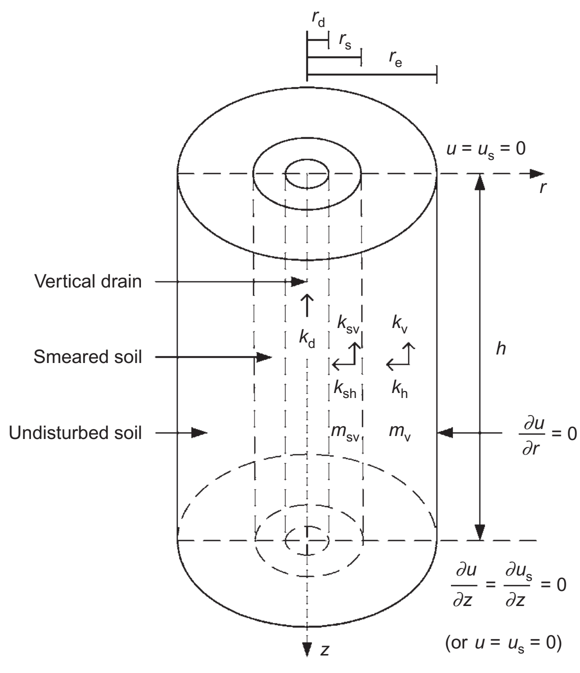

Fig. 1. A unit-cell consolidation model

the results from solutions with the equal-strain assumption are very close to those with the more realistic free-strain assumption (Barron, 1948; Hansbo, 1981; Onoue, 1988), as is also shown in this study (see Fig. 3, later). Equal strain is only a sufficient but not a necessary condition for deriving a solution for consolidation. From this point of view, different values of compressibility for the smeared and undisturbed soils may be used to reflect the consolidation under flexible loading conditions, in which there is no guarantee to maintain the strain compatibility. Strictly speaking, it should be acknowledged that a solution is simplified when derived from the governing equations (1) and (2) for consolidation under equal strain and one-dimensional compression conditions.

According to the continuity of the excess pore-water pressure and the flow rate at the interface between the vertical drain and the smeared soil, the drain resistance can be expressed as (Barron, 1948; Hansbo, 1981)

$$2k_{\mathrm{sh}}\left(\frac{\partial u_{\mathrm{s}}}{\partial r}\right)+r_{\mathrm{d}}k_{\mathrm{d}}\left(\frac{\partial^2 u_{\mathrm{s}}}{\partial z^2}\right)=0, \quad r=r_{\mathrm{d}} \tag{3}$$

where k_{d} is the hydraulic conductivity of the vertical drain.

The continuity of the excess pore-water pressure and the flow rate at the interface between the smeared soil and the undisturbed soil can be described by

$$u=u_{\mathrm{s}}, \quad r=r_{\mathrm{s}} \tag{4}$$

$$k_{\mathrm{h}}\left(\frac{\partial u}{\partial r}\right)=k_{\mathrm{sh}}\left(\frac{\partial u_{\mathrm{s}}}{\partial r}\right), \quad r=r_{\mathrm{s}} \tag{5}$$

The drainage boundary conditions can be expressed as follows

$$u=u_{\mathrm{s}}=0,\ z=0 \quad \text{for the pervious top} \tag{6}$$

$$u=u_{\mathrm{s}}=0,\ z=h \quad \text{for the pervious bottom} \tag{7}$$

$$\frac{\partial u}{\partial z}=\frac{\partial u_{\mathrm{s}}}{\partial z}=0,\ z=h \quad \text{for the impervious bottom} \tag{8}$$

$$\frac{\partial u}{\partial r}=0,\ r=r_{\mathrm{e}} \quad \text{for the impervious vertical boundary} \tag{9}$$

where h is the depth of the vertical drain.

The initial condition is given by

$$u=u_{\mathrm{s}}=\bar{u}=\bar{u}_{\mathrm{s}}=0,\ t=0 \tag{10}$$

The ten equations above describe the unit-cell consolidation problem to be solved.

THE ANALYTICAL SOLUTION

Figure 2 schematically shows the depth-varying increase in total stress in soil due to multi-ramp surcharge loading and unloading. To facilitate the derivation of the analytical solution, a new single equation is constructed to accurately describe the increase in total stress

$$\sigma(z,t)=\sum_{i=1}^{L}F_i(t)[\sigma_i(z)-\sigma_{i-1}(z)] \tag{11}$$

where

$$F_i(t)=\frac{t-t_{i,0}}{t_{i,1}-t_{i,0}}H\langle t-t_{i,0}\rangle[1-H\langle t-t_{i,1}\rangle]+H\langle t-t_{i,1}\rangle \tag{12}$$

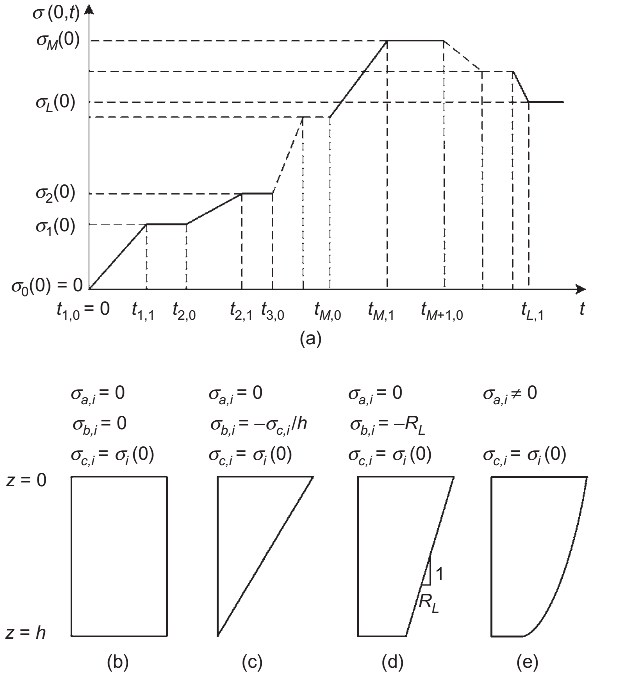

Fig. 2. Depth-varying and time-dependent increase in total stress in soil under multi-ramp loading and unloading: (a) multi-ramp loading and unloading; (b) rectangular; (c) triangular; (d) trapezoidal; (e) parabolic

$$\sigma_i(z)=\sigma_{a,i}z^2+\sigma_{b,i}z+\sigma_{c,i} \tag{13}$$

$$H\langle t-t_{i,j}\rangle=\begin{cases}0, & (t-t_{i,j})<0\\ 1, & (t-t_{i,j})\geq 0\end{cases}, \quad (j=0,1) \tag{14}$$

where $H\langle t-t_{i,j}\rangle$ is the Heaviside step function; L is the total number of loading and unloading ramps; $t_{i,0}$ and $t_{i,1}$ are the start time and end time of the ith ramp, respectively, as shown in Fig. 2(a); σ_i is the increase in total stress in soil at the end time of the ith ramp, and $\sigma_0=0$; and $\sigma_{a,i}$, $\sigma_{b,i}$ and $\sigma_{c,i}$ are coefficients describing the distribution of the increase in total stress as a function of depth. For rectangular, triangular and trapezoidal distributions, the values of $\sigma_{a,i}$, $\sigma_{b,i}$ and $\sigma_{c,i}$ can be

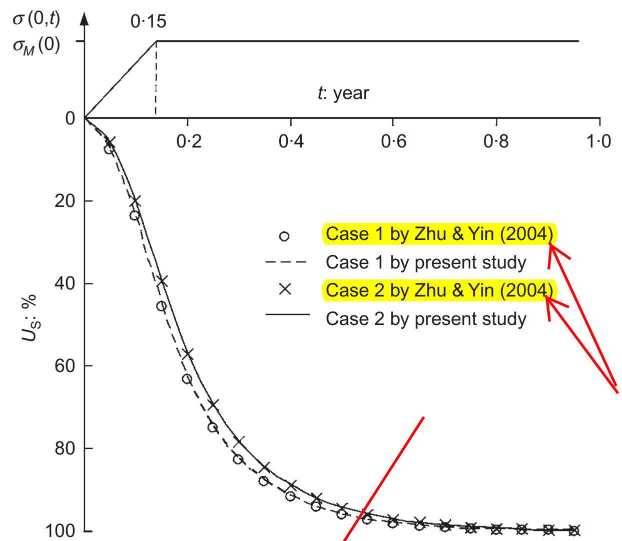

Fig. 3. A comparison between the solution proposed in the present study and that proposed by Zhu & Yin (2004)

readily derived from equation (13) according to the values of stress increase at $z=0$ and $z=h$, as presented in Figs 2(b)–2(d). For a parabolic distribution as shown in Fig. 2(e), $\sigma_{a,i}\neq 0$ and $\sigma_{c,i}=\sigma_i$ at $z=0$, and $\sigma_{b,i}$ can be specified.

By using the method of separation of variables and the Fourier series, Leo (2004) derived an analytical solution to approximate governing equations (see Table 1). These equations were simplified from equations (1) and (2) by ignoring the vertical flow and compression in the smear zone. A uniform increase in total stress in soil under single-ramp loading was considered. Lei & Jiang (2005) extended this solution to consider a depth-varying increase in total stress. In the present paper, the governing equations (1) and (2) are solved by adopting the derivation method and procedure of Leo (2004), as presented in detail in the Appendix. A depth-varying and time-dependent increase in total stress in soil under multi-ramp loading and unloading is considered, as given by equations (11) to (14). Analytical solutions are obtained for calculating the excess pore-water pressure at an arbitrary point in the undisturbed soil and the smeared soil, as given by

$$u=\frac{m_{\mathrm{v}}\gamma_{\mathrm{w}}}{k_{\mathrm{v}}}\sum_{n=1}^{\infty}\left\{[c_{1n}I_0(\mu_n r)+c_{2n}K_0(\mu_n r)+1]\times\frac{\sin(\omega_n z)}{\omega_n^2}\sum_{i=1}^{L}C_{n,i}(t)\right\} \tag{15}$$

$$u_{\mathrm{s}}=\frac{m_{\mathrm{sv}}\gamma_{\mathrm{w}}}{k_{\mathrm{sv}}}\sum_{n=1}^{\infty}\left\{[c_{3n}I_0(\mu_{sn} r)+c_{4n}K_0(\mu_{sn} r)+1]\times\frac{\sin(\omega_n z)}{\omega_n^2}\sum_{i=1}^{L}C_{n,i}(t)\right\} \tag{16}$$

where

$$\omega_n=\frac{(2n-1)\pi}{Dh} \tag{17}$$

$$C_{n,i}(t)=\frac{\sigma_{n,i}-\sigma_{n,i-1}}{t_{i,1}-t_{i,0}}\left[\mathrm{e}^{\frac{-8\left(T_{\mathrm{h}}-T_{\mathrm{h}i,1}\right)}{v_n}H\langle T_{\mathrm{h}}-T_{\mathrm{h}i,1}\rangle}-\mathrm{e}^{\frac{-8\left(T_{\mathrm{h}}-T_{\mathrm{h}i,0}\right)}{v_n}H\langle T_{\mathrm{h}}-T_{\mathrm{h}i,0}\rangle}\right] \tag{18}$$

where $D=1$ for pervious top and bottom boundaries, and its corresponding $\sigma_{n,i}$ is given by equation (28) in the Appendix; $D=2$ for pervious top and impervious bottom boundaries, and its corresponding $\sigma_{n,i}$ is given by equation (29); T_{h} is the time factor given by equation (45) in the Appendix; I_0 and K_0 are the modified Bessel functions of the first and second kind of zero order, respectively; the expressions for c_{1n}, c_{2n}, c_{3n}, c_{4n}, μ_n, μ_{sn}, $T_{\mathrm{h}i,j}$ and v_n are given by equations (82), (83), (71), (85), (38), (54), (70) and (46), respectively, in the Appendix.

As usual, the overall average degree of consolidation is defined in terms of the dissipation of excess pore-water pressure as a result of the maximum increase in total stress in soil as

$$U_{\mathrm{S}}(T_{\mathrm{h}})=\frac{\frac{1}{h}\int_0^h\sigma[z,t-(t-t_{M+1,0})H\langle t-t_{M+1,0}\rangle]\mathrm{d}z-\bar{u}_{\mathrm{o}}}{\frac{1}{h}\int_0^h\sigma_M(z,t_{M,1})\mathrm{d}z} \tag{19}$$

where σ_M is the maximum increase in total stress in soil at the end time $t_{M,1}$ of the Mth ramp of surcharge loading, as shown in Fig. 2(a). Based on equations (11) to (14), the following expressions can be derived:

$$\frac{1}{h}\int_0^h\sigma_M(z,t_{M,1})\mathrm{d}z=\frac{\sigma_{a,M}}{3}h^2+\frac{\sigma_{b,M}}{2}h+\sigma_{c,M} \tag{20}$$

$$\frac{1}{h}\int_0^h\sigma[z,t-(t-t_{M+1,0})H\langle t-t_{M+1,0}\rangle]\mathrm{d}z=\sum_{i=1}^{M}\left\{F_i(t)\left[\frac{\sigma_{a,i}-\sigma_{a,i-1}}{3}h^2+\frac{\sigma_{b,i}-\sigma_{b,i-1}}{2}h+\sigma_{c,i}-\sigma_{c,i-1}\right]\right\} \tag{21}$$

Based on equations (15) and (16), the overall average excess pore-water pressure can be derived as

$$\bar{u}_{\mathrm{o}}=\frac{\int_0^h\left[\int_{r_{\mathrm{s}}}^{r_{\mathrm{e}}}2\pi r u\mathrm{d}r+\int_{r_{\mathrm{d}}}^{r_{\mathrm{s}}}2\pi r u_{\mathrm{s}}\mathrm{d}r\right]\mathrm{d}z}{\pi(r_{\mathrm{e}}^2-r_{\mathrm{d}}^2)h}=\frac{2}{(r_{\mathrm{e}}^2-r_{\mathrm{d}}^2)Dh}\times\sum_{n=1}^{\infty}\left\{\frac{1}{\omega_n^3}\left(\frac{r_{\mathrm{e}}^2-r_{\mathrm{s}}^2}{k_{\mathrm{v}}}m_{\mathrm{v}}\gamma_{\mathrm{w}}\Omega_n+\frac{r_{\mathrm{s}}^2-r_{\mathrm{d}}^2}{k_{\mathrm{sv}}}m_{\mathrm{sv}}\gamma_{\mathrm{w}}\Omega_{sn}\right)\sum_{i=1}^{L}C_{n,i}(t)\right\} \tag{22}$$

where Ω_n and Ω_{sn} are given by equation (42) and equation (57), respectively, in the Appendix. Thus, by substituting equations (20) to (22) into equation (19), the overall average degree of consolidation can be obtained.

For easy use of the proposed solution, a simple Fortran program that solves the modified Bessel functions with freeware subroutines (Press *et al.*, 1992) has been developed. The results are obtained through double-precision arithmetic calculation.

VERIFICATION

In order to verify the validity and accuracy of the proposed analytical solution, the calculated results from the simplified cases of the proposed solution are compared with those from the analytical solutions of Zhu & Yin (2004) and Tang & Onitsuka (2000). A uniform increase in total stress in soil is considered. Zhu & Yin (2004) developed a solution to the more realistic free-strain consolidation under single-ramp loading. Drain resistance was ignored, and $k_{\mathrm{sv}}=k_{\mathrm{v}}$ and $m_{\mathrm{sv}}=m_{\mathrm{v}}$ were assumed (see Table 1). Tang & Onitsuka (2000) developed a solution to the equal-strain consolidation under multi-ramp loading. The solution was derived by assuming that vertical flow was governed by the average vertical hydraulic gradient, and that $k_{\mathrm{sv}}=k_{\mathrm{v}}$ and $m_{\mathrm{sv}}=m_{\mathrm{v}}$. For comparison purposes, the calculation parameters presented by Zhu & Yin (2004) and Tang & Onitsuka (2000) are adopted for the respective cases as Table 2 shows. Figures 3 and 4 show the comparisons of the calculated degrees of consolidation from the proposed solution with those from Zhu & Yin (2004) and Tang & Onitsuka (2000), respectively. It can be seen that excellent agreement is obtained.

For validation purposes, the proposed solution is also applied to a well-documented case study of test fill embankment at the Chek Lap Kok international airport in Hong Kong (Foott *et al.*, 1987; Handfelt *et al.*, 1987; Koutsoftas *et al.*, 1987; Koutsoftas, 1994; Koutsoftas & Cheung, 1994; Lo & Mesri, 1994). The test area was divided into four quadrants. By using a simplified finite-element method, Zhu *et al.* (2001) performed a detailed analysis of the consolidation behaviour of soils in the north-western quadrant of the test fill embankment. Soil parameters were selected from data provided by the original programme of site investigation and

Table 2. Calculation parameters

Identity	References	Drain properties	Soil properties	Drainage boundary, stress and loading conditions
Case 1	Zhu & Yin (2004)	$h = 2{\cdot}5$ m, $r_d = 0{\cdot}025$ m, $r_s = 0{\cdot}1$ m, $r_e = 0{\cdot}5$ m, $k_d = \infty$	$k_v/(m_v\gamma_w) = k_{sh}/(m_{sv}\gamma_w) = 1{\cdot}5$ m^2/year, $k_h/(m_v\gamma_w) = 3{\cdot}0$ m^2/year	$D = 2$, $L = M = 1$, $t_{1,1} = 0{\cdot}15$ year, $\sigma_{a,1} = \sigma_{b,1} = 0$, $\sigma_{c,1} =$ constant
Case 2	Zhu & Yin (2004)	As above, except $h = 10$ m	As above	As above
Case 3	Tang & Onitsuka (2000)	$h = 11{\cdot}2$ m, $r_d = 0{\cdot}035$ m, $r_s = 0{\cdot}07$ m, $r_e = 0{\cdot}7$ m, $k_d = 10^{-5}$m/s	$k_v = k_{sv} = k_h = 2\times10^{-9}$m/s, $k_{sh} = 2\times10^{-10}$m/s, $m_v = m_{sv} = 10^{-3}$kPa^{-1}	$D = 2$, $L = M = 2$, $\sigma_{a,i} = \sigma_{b,i} = 0$, $\sigma_{c,1} = 0{\cdot}5\sigma_{c,2}$, $\sigma_{c,2} =$ constant, $T_{h1,1} = 0{\cdot}5$, $T_{h2,0} = 3{\cdot}5$, $T_{h2,1} = 4{\cdot}0$
Case 4	Tang & Onitsuka (2000)	As above	As above	As above except $T_{h1,1} = 2{\cdot}0$, $T_{h2,0} = 4{\cdot}0$, $T_{h2,1} = 6{\cdot}0$
Case 5	Tang & Onitsuka (2000)	As above	As above	As above except $T_{h1,1} = 0{\cdot}5$, $T_{h2,0} = 6{\cdot}0$, $T_{h2,1} = 6{\cdot}5$
Case 6	Zhu *et al.* (2001)	$h = 6{\cdot}2$ m, $r_d = 0{\cdot}02745$ m, $r_s = 0{\cdot}137$ m, $r_e = 0{\cdot}7875$ m, $k_d = 1{\cdot}389\times10^{-5}$m/s	$k_v = k_{sv} = k_{sh} = 2{\cdot}2\times10^{-9}$m/s, $k_h = 4{\cdot}0\times10^{-9}$m/s, $m_v = m_{sv} = 1{\cdot}183\times10^{-3}$kPa^{-1}	$D = 2$, $L = M = 3$, $\sigma_{a,i} = \sigma_{b,i} = 0$, $\sigma_{c,1} = 51{\cdot}8$ kPa, $\sigma_{c,2} = 151{\cdot}8$ kPa, $\sigma_{c,3} = 255{\cdot}9$ kPa, $t_{1,1} = 17$ days, $t_{2,0} = 77$ days, $t_{2,1} = 95$ days, $t_{3,0} = 274$ days, $t_{3,1} = 299$ days
Case 7	Present study	$h = 15$ m, $r_d = 0{\cdot}035$ m, $r_s = 0{\cdot}14$ m, $r_e = 0{\cdot}5$ m, $k_d = 10^{-5}$m/s	$k_v = 10^{-9}$m/s, $k_h = 2\times10^{-9}$m/s, $k_{sv} = k_v$, $k_{sh} = 2\times10^{-10}$m/s, $m_v = 10^{-3}$kPa^{-1}, $m_{sv} = m_v$	$D = 2$, $\sigma_{b,1} = -\sigma_{c,1}/(2h)$ for trapezoidal distribution, $\sigma_{a,1} = -\sigma_{c,1}/(2h^2)$ and $\sigma_{b,1} = 0$ for parabolic distribution

laboratory testing. In the present study, the consolidation behaviour of upper marine clay was analysed. Figure 5(a) shows the vertical locations of pneumatic piezometers (PP60 and PP43) and subsurface settlement anchors (A1~A5), which were placed at the centre of a triangular grid (in plan) of prefabricated vertical drains. The field data measured by the settlement anchors reveal that the contribution of the compression of upper marine clay layer to the total ground surface settlement exceeds 70%. All the calculation parameters in case 6 in Table 2 are adopted from Zhu *et al.* (2001), except for the coefficient of volume compressibility of soil. This coefficient was back-calculated from the compression of the upper marine clay layer, which was derived by subtracting the ultimate settlement measured by A5 from that measured by A1 as they are reported by Zhu *et al.* (2001). Drain resistance is considered by adopting the permeability of vertical drain $k_d = 10^{-5}$ m/s from Zhu *et al.* (2001). To investigate the effect of drain resistance on consolidation, an extremely large value of $k_d = 1$ m/s is also employed to obtain the results for consolidation without drain resistance. By using the proposed solution, namely, equation (19), the degree of consolidation $U_S(t)$ was calculated. The compression of each soil segment $S_i(t)$ between one of the settlement anchors A1~A4 and the bottom anchor A5 was derived by $S_i(t) = m_v\sigma_M h_i U_S(t)$, where σ_M is the maximum increase in total stress and h_i is the thickness of soil segment. The excess pore-water pressures at the measurement positions of PP60 and PP43 were also calculated by using equation (15). Fig. 5(b) shows comparisons of calculated compressions and measured compressions using data obtained from settlement anchors A1 to A5. Fig. 5(c) shows comparisons of calculated excess pore-water pressures with measured ones. The solid lines represent the calculated results with drain resistance. It can be seen that these results are in reasonably good agreement with the field data.

The dashed lines in Figs 5(b) and 5(c) represent the calculated results without drain resistance. It can be observed that the rate of compression is generally overestimated, especially during a short period after a ramp load is applied. During the holding period of the maximum applied load, the rate of compression is relatively significantly overestimated. Moreover, the rate of excess pore-water pressure dissipation is exaggeratedly overestimated, especially during the holding periods of the applied loads. This will lead to the design of a loading rate that tends to be unsafe. It is evident that drain resistance has a significant influence on consolidation, and it plays an important role in the consolidation analysis.

MULTI-RAMP LOADING EFFECTS AND A NEW DEGREE OF DISSIPATION

It can be seen from Fig. 4 that the loading path over time (for a given load increment) has a significant effect on the rate of consolidation. To investigate this effect in depth, the consolidation under single-ramp loading along different paths is analysed. A uniform increase in total stress in soil is considered. In the present and subsequent analyses, the calculation parameters in case 7 in Table 2 are assumed unless otherwise stated. The consolidation under instantaneous loading is also considered by simply letting $t_{1,1} = t_{1,0}$ and $L = 1$ (see Fig. 2(a) or equation (11)). Fig. 6(a) shows the calculated degrees of consolidation U_S based on the conventional definition given by equation (19). The solid circles represent the results for the consolidation subjected to the same load along different paths. From the slopes of the loading paths and their corresponding consolidation curves, it can be seen that the shorter the loading path, the higher the consolidation rate. Moreover, in the same loading period, the lower the loading rate, the lower the degree of consolidation, as is for example shown by the points p and q and their corresponding points P and Q in Fig. 6(a). Clearly, the conventional definition of degree of consolidation is closely related to ground surface settlement. It is commonly used to estimate the time it takes for surcharge preloading to achieve a target degree of consolidation, say 90%, relative to the ultimate ground surface settlement caused by the preloading itself. Fig. 6(a) implies that, as the loading rate needed to reach the same load increases, the settlement rate will increase and the construction time will decrease, as is expected.

On the other hand, the loading rate that can be applied in practice is controlled by the stability of surcharge fills (Ladd, 1991; Indraratna *et al.*, 2005c; Rujikiatkamjorn & Indraratna, 2009). Specifically, it depends on an increase in shear strength due to consolidation. In this sense, one would expect that as the loading rate needed to reach a given load decreases, the degree of dissipation of excess pore-water pressure relative to the consolidation stress induced by an applied load will increase. Accordingly, the stability of surcharge fills should improve. The effect of consolidation on shear strength should not be ignored in the determination of a safe and cost-effective loading rate. To account for this effect in preliminary design of vertical drains and surcharge fills, the assessment of the stability of surcharge fills is usually carried out based on the gain in undrained strength of soil by $\Delta s_u = \alpha\Delta\sigma'_v = \alpha\sigma_M U_S$ (Li & Rowe, 2001; Bergado *et al.*, 2002; Rujikiatkamjorn & Indraratna, 2009; Sinha *et al.*, 2009; Chai & Duy, 2013), where α is the undrained strength gain ratio, which is almost constant for a given normally consolidated soil (Ladd & Foott, 1974; Mesri, 1989; Wang *et al.*, 2008); $\Delta\sigma'_v$ is the increase in effective vertical stress; and σ_M is the maximum increase in total stress due to surcharge loading. Since the value of U_S at a given point in time increases with an increase in loading rate (as is shown by Fig. 6(a)), it follows that Δs_u should also increase with an increase in loading rate. This would, however, result in the stability of surcharge fills improving as the loading rate increases. Therefore, under ramp loading conditions, it becomes unreasonable to use the degree of consolidation U_S to estimate the gain in undrained strength. For this reason, it would be better to use $\Delta\sigma'_v = \sigma(t)[1 - u(t)/\sigma(t)]$ to estimate the gain in undrained strength and the stability of surcharge fills,

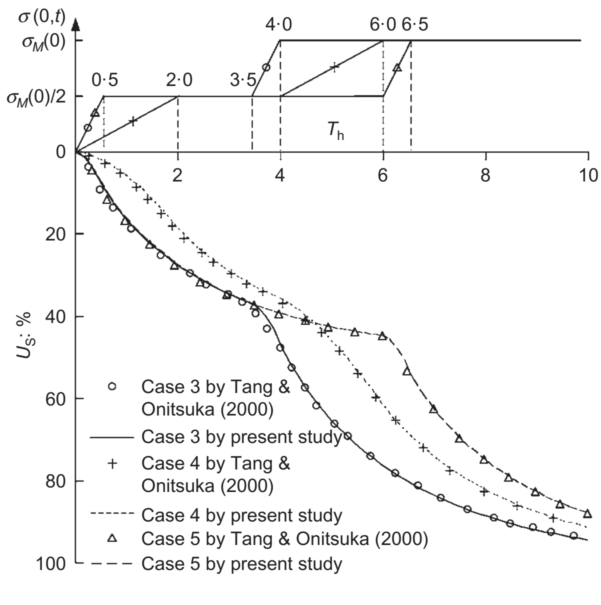

Fig. 4. A comparison between the solution proposed in this study and that proposed by Tang & Onitsuka (2000)

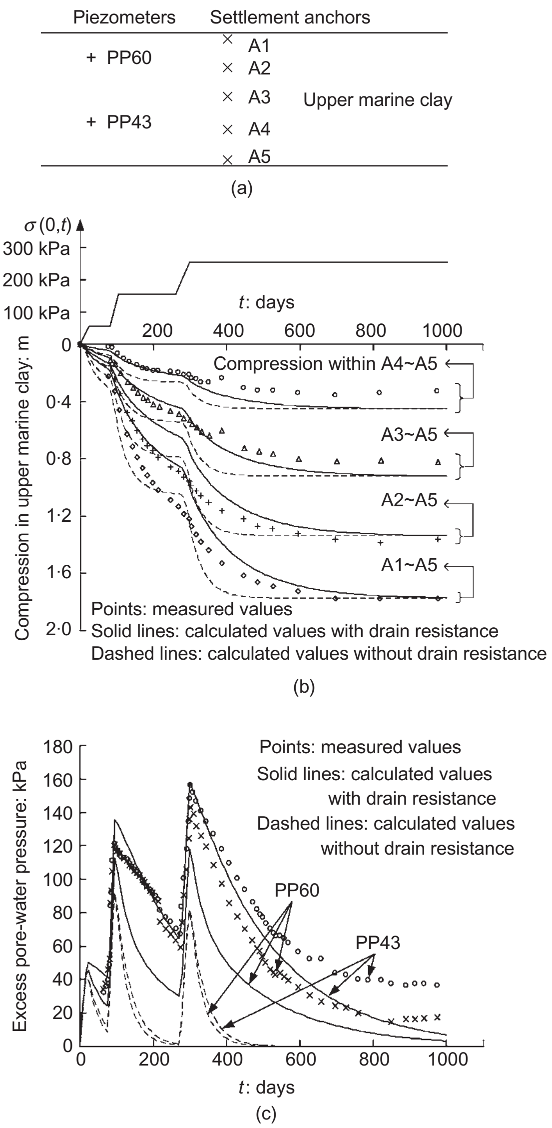

Fig. 5. Comparisons between calculated values and field measurements (measured data reproduced from Zhu *et al.* (2001)): (a) instrumentation; (b) compression in upper marine clay; (c) excess pore-water pressure at measurement positions PP60 and PP43

where $\sigma(t)$ and $u(t)$ are an increase in total stress and excess pore-water pressure at any given point in time, respectively. To quantify $[1-u(t)/\sigma(t)]$, a degree of dissipation is thus defined in terms of dissipation of excess pore-water pressure as a result of an increase in total stress induced at that point in time as follows

$$U_P(T_h) = 1 - \frac{\max(\bar{u}_o, 0)}{\dfrac{1}{h}\displaystyle\int_0^h \sigma(z,t)\mathrm{d}z} \tag{23}$$

where 'max' means taking the maximum value between 0 and the overall average excess pore-water pressure $\bar{u}_o$, in case $\bar{u}_o$ becomes negative under unloading; and

$$\frac{1}{h}\int_0^h \sigma(z,t)\mathrm{d}z = \sum_{i=1}^{L}\left\{F_i(t)\left[\frac{\sigma_{a,i}-\sigma_{a,i-1}}{3}h^2 + \frac{\sigma_{b,i}-\sigma_{b,i-1}}{2}h + \sigma_{c,i}-\sigma_{c,i-1}\right]\right\} \tag{24}$$

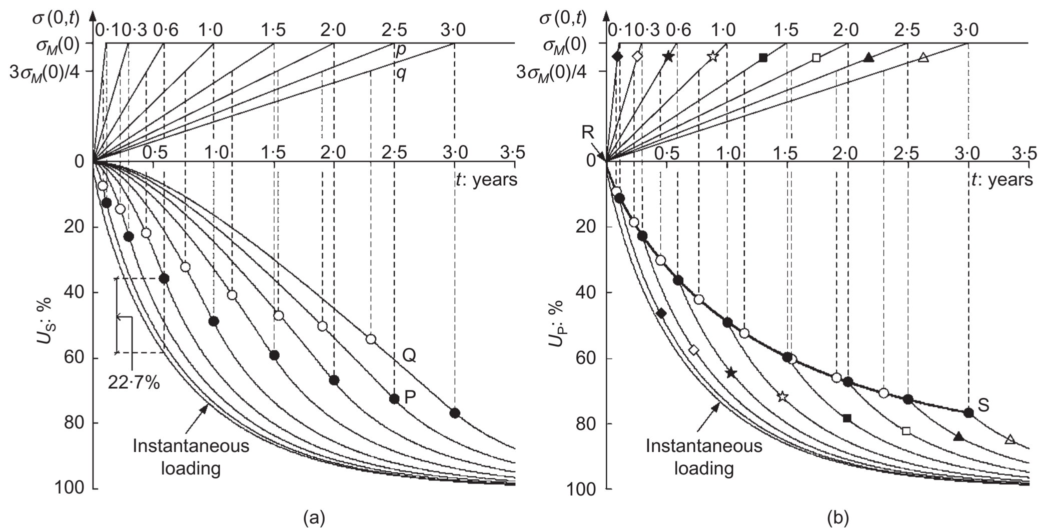

Fig. 6. Effects of loading path over time on consolidation: (a) U_S plotted against time; (b) U_P plotted against time

Table 3. Difference between U_S and U_P for consolidation under three-fourths of the same load applied at different rates

Duration of ramp loading: years	0·1	0·3	0·6	1·0	1·5	2·0	2·5	3·0
U_S: %	6·4	13·9	22·6	31·1	38·9	44·7	49·1	52·6
U_P: %	8·5	18·6	30·1	41·5	51·9	59·6	65·5	70·1
Difference: %	2·1	4·7	7·5	10·4	13·0	14·9	16·4	17·5

It can be easily proved that only during the holding period of the maximum surcharge load, namely from $t = t_{M,1}$ to $t_{M+1,0}$ (see Fig. 2(a)), are results from equations (19) and (23) the same. Based on the degree of dissipation given by equation (23), the gain in undrained strength of normally consolidated soil due to consolidation can be estimated as follows

$$\Delta s_u = \alpha \Delta \sigma'_v = \alpha \sigma(t) U_P \quad (25)$$

Under ramp loading conditions, equation (25) should be used to assess the gain in undrained strength of soils under the centre-line of surcharge fills. Based on equation (25), the gain in undrained strength along a potential failure surface can be derived by using an approximate method proposed by Li & Rowe (2001). On this basis, the 'stress history and normalised soil engineering properties' (Shansep) technique (Ladd & Foott, 1974; Ladd, 1991) can be used to predict the stability of surcharge fills for preliminary design purposes. This is, however, beyond the scope of this study. For details of the Shansep technique of Ladd & Foott (1974) and the method of Li & Rowe (2001), see the source references.

Figure 6(b) shows the calculated degree of dissipation U_P in relation to the stability of surcharge fills based on equation (23). The open circles represent the results when 75% of the same load is applied at different rates. It can be seen that the lower the loading rate required to reach the same load, the higher the degree of dissipation. The open circles in Fig. 6(a) represent the corresponding results of degree of consolidation U_S, based on the conventional definition given by equation (19). Table 3 compares the values of U_S and U_P shown in Fig. 6(a) and Fig. 6(b). It can be observed that the difference between U_S and U_P increases with decreasing loading rate (i.e. increasing duration of ramp loading). The conventional definition of degree of consolidation in relation to ground surface settlement underestimates the degree of excess pore-water pressure dissipation in relation to the stability of surcharge fills. Therefore, using the conventional definition of degree of consolidation to assess the gain in shear strength and the stability of surcharge fills will result in a loading rate that is over conservative and a loading process that is very time consuming.

From Fig. 6(b), it can also be seen that each curve of U_P consists of two segments – a segment for the consolidation during the ramp-loading period and a segment for the consolidation during the holding period of the applied load. For the relevant calculation parameters, during the ramp-loading period, all the curve segments of U_P for different loading rates coincide on the same curve, as shown by the curve RS in Fig. 6(b). This unique feature can be utilised to efficiently determine the loading rate by simply reading off, from that curve, the duration of ramp loading for a required degree of dissipation to ensure the stability of surcharge fills. Moreover, Fig. 6 shows that the degree of consolidation and the degree of dissipation under time-dependent loading, as encountered in most practical situations, may be significantly overestimated if an instantaneous loading condition is assumed. For example, the degree of consolidation at a time point of 0·6 years is 36·2% if the duration of ramp loading is 0·6 years. The degree of consolidation at the same point in time for instantaneous loading is 58·9%. The difference exceeds 20%. It is evident that the instantaneous loading assumption will lead to the design of a loading rate that tends to be unsafe and the design of a drain spacing that tends to be ineffective.

Figure 7 shows the calculated degree of consolidation and degree of dissipation under three loading conditions with the same duration of ramp loading but different maximum loads. It can be seen that the same curve of U_S or U_P is obtained for loading at different rates. If the applied loads are normalised by their respective maximum loads, the same loading path over time will be obtained. Fig. 7 illustrates that the degree of consolidation and the degree of dissipation are, by definition, independent of the actual loading rate. They are dependent on the loading path over time or the loading rate normalised

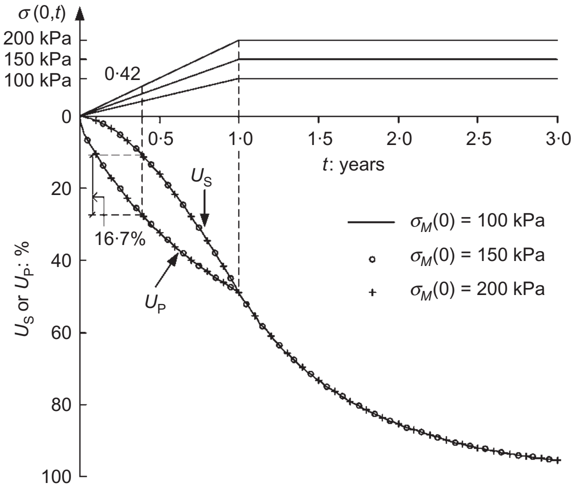

Fig. 7. Effects of loading rate on consolidation

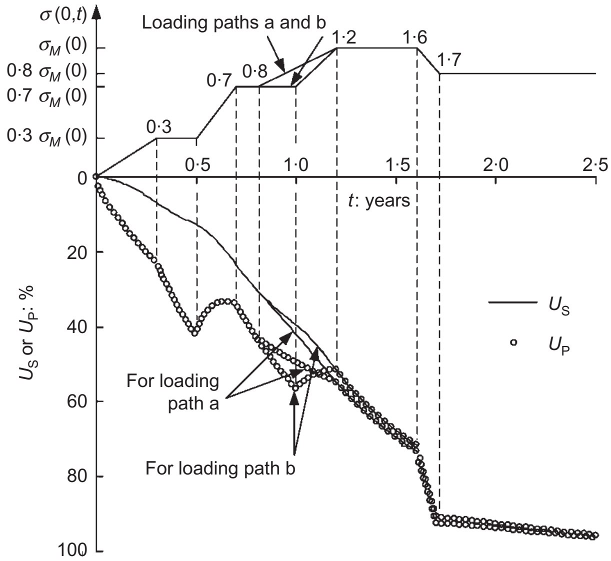

Fig. 9. Consolidation under multi-ramp loading and unloading

by the maximum load, as Fig. 6 also shows. In the mathematical sense, the degree of consolidation and the degree of dissipation are a function of $[\partial(\sigma/\sigma_M)/\partial t]$ instead of $(\partial\sigma/\partial t)$.

For the loading conditions assumed in Fig. 7, the normalised loading rate is 1·0/year. The maximum difference between U_S and U_P is 16·7%. Fig. 8 shows the maximum differences between U_S and U_P for consolidation under loading at different normalised rates. The maximum difference increases almost linearly with the increase in the normalised loading rate. It should be noted that the larger the normalised loading rate, the longer the loading path over time, and the lower the actual loading rate required to reach the same load. Therefore, Fig. 8 demonstrates that the maximum difference between U_S and U_P increases as the actual loading rate needed to reach the same load decreases.

Figure 9 shows an analysis of the consolidation under multi-ramp loading and unloading. Two loading paths denoted by 'a' and 'b' are considered. It can be seen that the degree of consolidation in relation to ground surface settlement U_S increases monotonically with time. Applying the load at an earlier time leads to a higher degree of consolidation U_S, as revealed by comparing the consolidation curves of U_S for the two loading paths. However, the degree of dissipation in relation to the stability of surcharge fills U_P may be reversed during a subsequent loading. This takes place when the rate of subsequent loading is high enough for building up the excess pore-water pressure (see the curves of U_P for the second stage of loading paths 'a' and 'b'). It also occurs when the holding period of the previously applied load is long enough for the induced excess pore-water pressure to be dissipated (see the curve of U_P for the third stage of loading path 'b'). Nevertheless, before the next load is applied, the longer the holding period of the previous load, the higher the degree of dissipation, as is expected.

EFFECTS OF STRESS DISTRIBUTION AS A FUNCTION OF DEPTH

Figure 10 shows the calculated degree of consolidation U_S and degree of dissipation U_P under single-ramp loading at a normalised rate of 1·0/year. Four different distributions of the increase in total stress in soil as a function of depth are considered. For the trapezoidal and parabolic distributions of stress, the increase in total stress at the bottom plane of the vertical drain is assumed to be half of that at the top plane. The calculation parameters are given in Fig. 2 and case 7 of Table 2. It can be seen from Fig. 10 that the lower and upper bounds to U_S or U_P are given by the curves

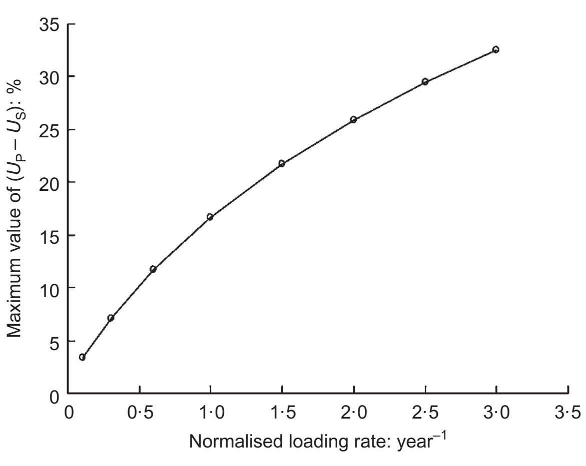

Fig. 8. Maximum differences between U_S and U_P under loading at different normalised rates

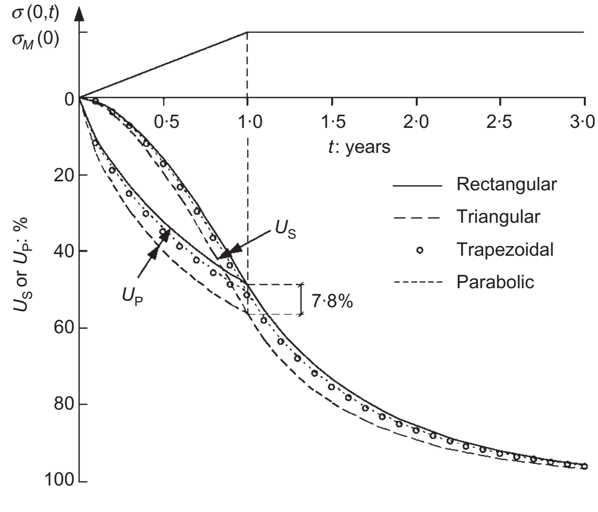

Fig. 10. Effects of total stress distribution as a function of depth on consolidation

with rectangular and triangular stress distributions, respectively. The maximum difference in the degree of consolidation or the degree of dissipation between these two extreme cases is 7·8%. However, the degree of consolidation and the degree of dissipation for a more realistic trapezoidal or parabolic stress distribution are only very slightly underestimated if a uniform rectangular stress distribution is assumed. In contrast to the effect of loading path over time, the effect of stress distribution as a function of depth on consolidation is insignificant and may be neglected.

SMEAR EFFECTS

The effect of the horizontal hydraulic conductivity of smeared soil on consolidation has been the subject of many investigations. However, in the existing solutions, the vertical hydraulic conductivity and the volume compressibility of smeared soil are either assumed to be unchanged from the undisturbed state or ignored. For this reason, their effects on consolidation are analysed in this section. The soil is subjected to a uniform increase in total stress under single-ramp loading at a normalised rate of 1·0/year.

Vertical hydraulic conductivity

Figures 11(a) and 11(b) show the calculated degree of consolidation U_S and degree of dissipation U_P, respectively. The horizontal hydraulic conductivity of undisturbed soil k_h is assumed to take the same value as the vertical one, namely k_v. This assumption is made to eliminate any possible interference from their difference. Three different values of drain spacing, designated by the radius of the effective influence zone of the vertical drain, r_e, are also assumed. The radius of the smeared zone is assumed fixed. Thus, the smaller is the value of r_e, the larger is the share of smeared soil in the effective influence zone. It can be seen that the vertical hydraulic conductivity of smeared soil k_{sv} has almost no effect on consolidation. This implies that the consolidation is mostly governed by radial flow. This is unsurprising, given that the flow path in the radial direction is substantially shorter than that in the vertical direction, and the degree of consolidation is inversely proportional to the square of flow path.

Volume compressibility

In the smear zone, reduced volume compressibility may be expected following the disturbance of the soil structure (Burland, 1990). This may also be inferred from the experimentally observed reduction in the void ratio and the water content of smeared soil (Indraratna & Redana, 1998; Hird & Moseley, 2000; Sharma & Xiao, 2000; Sathananthan & Indraratna, 2006; Weber *et al.*, 2010; Juneja *et al.*, 2013; Rujikiatkamjorn *et al.*, 2013). Direct experimental results of the volume compressibility of smeared soil, as most recently reported by Rujikiatkamjorn *et al.* (2013), confirm a reduction. Figure 12 shows the effects of the volume compressibility of smeared soil m_{sv} on the calculated degree of consolidation U_S and the degree of dissipation U_P. It is evident that U_S and U_P are slightly underestimated if the smeared soil is assumed to have the same volume compressibility as the undisturbed soil (i.e. $m_{sv} = m_v$). In general, a shorter drain spacing or a larger share of smeared soil in the effective influence zone leads to a relatively significant effect of the volume compressibility of smeared soil on consolidation. This verifies that both the volume compressibility and the hydraulic conductivity play important roles in the consolidation of heterogeneous soil, and that they cannot be represented by a single coefficient of consolidation (Lee *et al.*, 1992; Pyrah, 1996; Zhu & Yin, 1999; Huang *et al.*, 2010). From a quantitative point of view, the volume compressibility of smeared soil has only a negligible influence on the degree of consolidation U_S in relation to ground surface settlement. It also has a limited influence on the degree of dissipation U_P in relation to the stability of surcharge fills when the radius of the effective influence zone of the vertical drain r_e is greater than 0·3 m, which corresponds to a drain spacing of about 0·6 m. The maximum difference in U_P between the cases with $m_{sv} = 0{\cdot}5m_v$ and $m_{sv} = m_v$ is only 6·7% when $r_e = 0{\cdot}2$ m.

Figure 13 shows the effects of the volume compressibility of smeared soil on the calculated degree of consolidation and degree of dissipation, when the vertical drain is a sand or stone column with a large diameter. The radii of the drain and its effective influence zone are assumed to be 0·3 m and 0·9 m, respectively. According to Weber *et al.* (2010), the radius of the smear zone is assumed to be 2·5 times the radius of the drain, that is, $r_s = 0{\cdot}75$ m. The arching effect due to load transfer at the interface between the column and smeared soil (Indraratna *et al.*, 2013) is ignored. The calculated results may be considered as a first approximation to the analysis of consolidation with a sand or stone column. By comparing Fig. 13 with Fig. 12, it can be seen that the effect of the volume compressibility of smeared soil on consolidation with a large-diameter sand or stone column is more significant than that with a small-diameter prefabricated drain. However, the maximum differences in the degree of consolidation and the degree of dissipation between the cases with $m_{sv} = 0{\cdot}5m_v$ and $m_{sv} = m_v$ are only 7·3% and 8·3%, respectively.

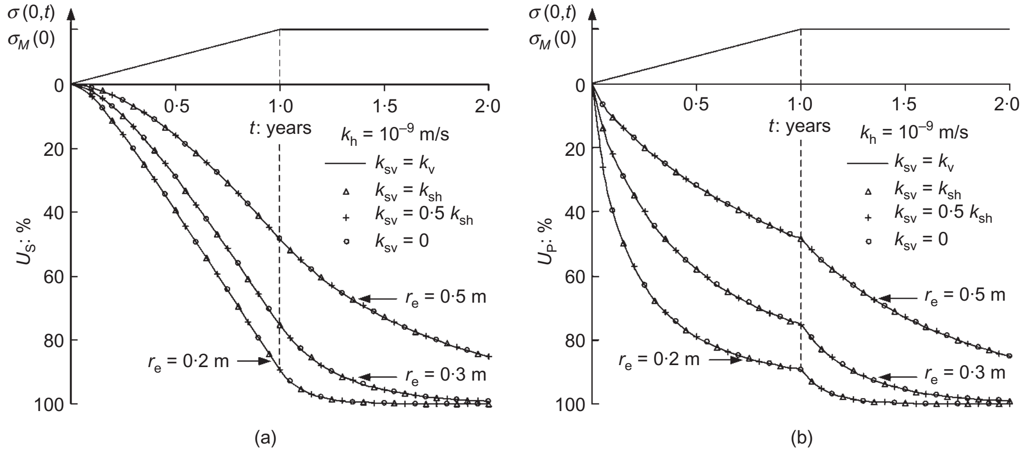

Fig. 11. Effects of the vertical hydraulic conductivity of smeared soil on consolidation: (a) U_S plotted against time; (b) U_P plotted against time

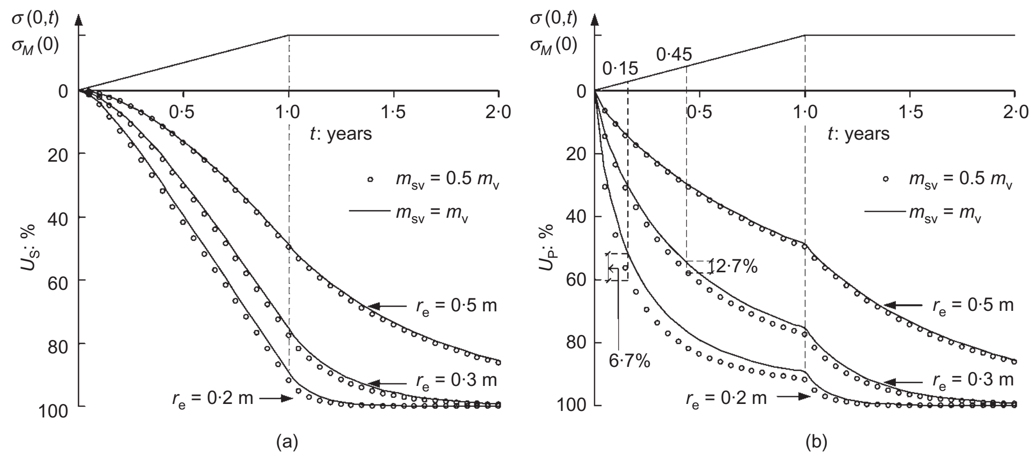

Fig. 12. Effects of the volume compressibility of smeared soil on consolidation with a prefabricated vertical drain: (a) U_S plotted against time; (b) U_P plotted against time

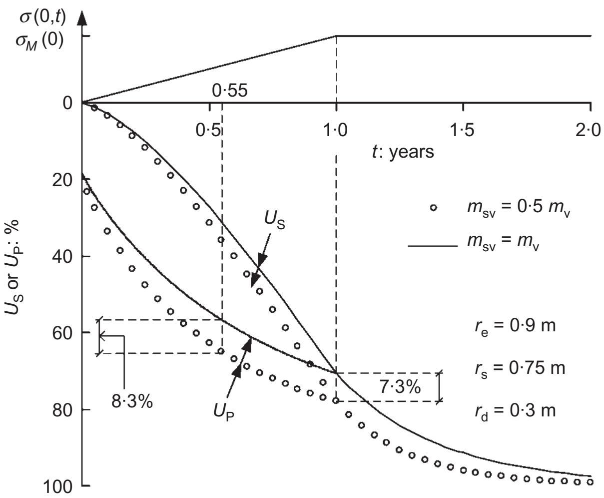

Fig. 13. Effects of the volume compressibility of smeared soil on consolidation with a large-diameter sand or stone column

SUMMARY AND CONCLUSIONS

A simplified analytical solution is proposed for a unit-cell model of equal-strain consolidation with a vertical drain under multi-ramp surcharge loading. The drain and soil are assumed to have uniform and constant material properties. The increase in total stress in soil due to loading is assumed to vary with both time and depth. The solution is derived from generalised governing equations with simultaneous vertical and radial flows. The combined effects of drain resistance and smear are also fully taken into account. The conventional definition of degree of consolidation is interpreted in relation to the ultimate ground surface settlement due to surcharge loading. A new degree of dissipation of excess pore-water pressure is defined and interpreted in relation to the stability of surcharge fills. Based on the calculated results, the following conclusions can be drawn.

(*a*) During the ramp-loading period, the conventional definition of degree of consolidation interpreted in relation to ground surface settlement underestimates the degree of dissipation in relation to the stability of surcharge fills. The maximum difference between them increases almost linearly with the decrease in the loading rate required to reach the same load. For the parameters considered, the maximum difference increases from 11·7% to 16·7% as the duration of ramp loading increases from 0·6 to 1 year. The reason for this is that, as the loading rate required to reach the same load reduces, the settlement rate will go down, but the dissipation rate of excess pore-water pressure relative to currently induced increase in total stress in the soil will go up. Using the conventional definition of degree of consolidation to assess the gain in soil strength and the stability of surcharge fills will result in a loading process that is time consuming.

(*b*) Based on the degree of dissipation, a new equation is proposed to estimate the gain in undrained strength due to consolidation for assessing the stability of surcharge fills.

(*c*) The degree of consolidation and the degree of dissipation are, by definition, governed by the loading path over time or the loading rate normalised by the maximum load, instead of the actual loading rate. Compared with the effect of loading path over time, the effect of stress distribution as a function of depth on consolidation is insignificant and may be neglected.

(*d*) The degree of consolidation and the degree of dissipation are overestimated if an instantaneous loading condition is assumed. For the parameters considered, the overestimation can exceed 20% if the duration of the ramp loading phase of a surcharge load is 0·6 years. The instantaneous loading assumption will lead to the design of a loading rate that tends to be unsafe and the design of a drain spacing that tends to be ineffective.

(*e*) The degree of consolidation and the degree of dissipation are slightly underestimated if the smeared soil is assumed to have the same volume compressibility as the undisturbed soil. For the parameters considered, the maximum underestimation is not over 8.3% when the volume compressibility of smeared soil is halved.

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APPENDIX

DERIVATION PROCEDURES

Loading and unloading

By introducing the Fourier sine series, equation (11) can be expressed as

$$\sigma(z,t) = \sum_{n=1}^{\infty}\left\{\sum_{i=1}^{L}\left[F_i(t)\left(\sigma_{n,i}-\sigma_{n,i-1}\right)\right]\right\}\sin(\omega_n z) \tag{26}$$

where ω_n is dependent on an indicator parameter (D) of the drainage boundary, as given by equation (17); $\sigma_{n,i}$ is the corresponding Fourier coefficient and is calculated as

$$\sigma_{n,i} = \frac{2}{h}\int_0^h \sigma_i \sin(\omega_n z)\mathrm{d}z \tag{27}$$

Substituting equation (13) into equation (27) gives

$$\sigma_{n,i} = \frac{2}{\omega_n^3 h}\left[-4\sigma_{a,i}+\sigma_{a,i}\omega_n^2 h^2+\sigma_{b,i}\omega_n^2 h+2\sigma_{c,i}\omega_n^2\right], \quad \text{for } D=1 \tag{28}$$

$$\sigma_{n,i} = \frac{2}{\omega_n^3 h}\left[-2\sigma_{a,i}+2\sigma_{a,i}\omega_n h(-1)^{n-1} + \sigma_{b,i}\omega_n(-1)^{n-1}+\sigma_{c,i}\omega_n^2\right], \quad \text{for } D=2 \tag{29}$$

Consolidation of the undisturbed soil

Again by introducing the Fourier sine series, the excess pore-water pressure of the undisturbed soil can be expressed in fulfilment of equations (6) to (8) of the top and bottom drainage boundary conditions as follows

$$u(r,z,t) = \sum_{n=1}^{\infty} u_n(r,t)\sin(\omega_n z) \tag{30}$$

$$\bar{u}(z,t) = \sum_{n=1}^{\infty} \bar{u}_n(t)\sin(\omega_n z) \tag{31}$$

where u_n and $\bar{u}_n$ are their corresponding Fourier coefficients.

Substituting equations (30), (31) and (26) along with equation (12) into the governing equation (1) yields

$$\frac{k_\mathrm{h}}{\gamma_\mathrm{w}}\left[\frac{\partial^2 u_n(r,t)}{\partial r^2}+\frac{1}{r}\frac{\partial u_n(r,t)}{\partial r}\right]-\frac{k_\mathrm{v}}{\gamma_\mathrm{w}}\omega_n^2 u_n(r,t) = -m_\mathrm{v}\left[f_n-\frac{\partial \bar{u}_n(t)}{\partial t}\right] \tag{32}$$

$$f_n = \sum_{i=1}^{L}\frac{1-H\langle t-t_{i,1}\rangle}{t_{i,1}-t_{i,0}}H\langle t-t_{i,0}\rangle\left(\sigma_{n,i}-\sigma_{n,i-1}\right) \tag{33}$$

By using the method of separation of variables, the following equation can be written

$$u_n(r,t) = A_n(r)B_n(t) \tag{34}$$

where A and B are functions of radial coordinate and time, respectively. Substituting equation (34) into equation (32) gives

$$\frac{k_\mathrm{h}}{\gamma_\mathrm{w}}\left[\frac{\partial^2 A_n(r)}{\partial r^2}+\frac{1}{r}\frac{\partial A_n(r)}{\partial r}\right]-\frac{k_\mathrm{v}}{\gamma_\mathrm{w}}\omega_n^2 A_n(r) = -\frac{m_\mathrm{v}}{B_n(t)}\left[f_n-\frac{\partial \bar{u}_n(t)}{\partial t}\right] = -\lambda_n \tag{35}$$

where λ_n is the separation constant. A solution to equation (35) is

$$A_n(r) = \lambda_n\phi_n\left[c_{1n}I_0(\mu_n r)+c_{2n}K_0(\mu_n r)+1\right] \tag{36}$$

where I_0 and K_0 are the modified Bessel functions of the first and second kind of zero order, respectively; c_{1n} and c_{2n} are the constants of integration to be determined; and

$$\phi_n = \frac{\gamma_\mathrm{w}}{k_\mathrm{v}\omega_n^2} \tag{37}$$

$$\mu_n^2 = \frac{k_\mathrm{v}\omega_n^2}{k_\mathrm{h}} \tag{38}$$

The average excess pore-water pressure at a given depth is

$$\bar{u}(z,t) = \frac{1}{\pi\left(r_\mathrm{e}^2-r_\mathrm{s}^2\right)}\int_{r_\mathrm{s}}^{r_\mathrm{e}} u(r,z,t)2\pi r\mathrm{d}r = \frac{1}{\pi\left(r_\mathrm{e}^2-r_\mathrm{s}^2\right)}\sum_{n=1}^{\infty}\left[\int_{r_\mathrm{s}}^{r_\mathrm{e}} A_n(r)2\pi r\mathrm{d}r\right]B_n(t)\sin(\omega_n z) \tag{39}$$

From equations (31) and (39), the following can be derived

$$\bar{u}_n(t) = \frac{B_n(t)}{\pi\left(r_\mathrm{e}^2-r_\mathrm{s}^2\right)}\int_{r_\mathrm{s}}^{r_\mathrm{e}} A_n(r)2\pi r\mathrm{d}r \tag{40}$$

Substituting equation (36) into equation (40) yields

$$\bar{u}_n(t) = \lambda_n\phi_n\Omega_n B_n(t) \tag{41}$$

where

$$\Omega_n = 1+\frac{2c_{1n}\left[\mu_n r_\mathrm{e}I_1(\mu_n r_\mathrm{e})-\mu_n r_\mathrm{s}I_1(\mu_n r_\mathrm{s})\right]-2c_{2n}\left[\mu_n r_\mathrm{e}K_1(\mu_n r_\mathrm{e})-\mu_n r_\mathrm{s}K_1(\mu_n r_\mathrm{s})\right]}{(\mu_n r_\mathrm{e})^2-(\mu_n r_\mathrm{s})^2} \tag{42}$$

where I_1 and K_1 are the modified Bessel functions of the first and second kind of order one, respectively.

Substituting equation (41) into equation (35) yields

$$m_\mathrm{v}\left[f_n-\lambda_n\phi_n\Omega_n\frac{\partial B_n(t)}{\partial t}\right] = \lambda_n B_n(t) \tag{43}$$

A solution of equation (43) is

$$B_n(t) = \frac{1}{\lambda_n\phi_n}\left(a_n\mathrm{e}^{\frac{-8T_\mathrm{h}}{v_n}}+m_\mathrm{v}\phi_n f_n\right) \tag{44}$$

where a_n is the constant of integration to be determined; T_h is the time factor; and

$$T_\mathrm{h} = \frac{k_\mathrm{h}t}{m_\mathrm{v}\gamma_\mathrm{w}(2r_\mathrm{e})^2} \tag{45}$$

$$v_n = \frac{2\Omega_n}{(\mu_n r_\mathrm{e})^2} \tag{46}$$

Based on equations (34), (36) and (44), equation (30) can be rewritten as

$$u(r,z,t) = \sum_{n=1}^{\infty}\left[c_{1n}I_0(\mu_n r)+c_{2n}K_0(\mu_n r)+1\right] \times\left[a_n\mathrm{e}^{\frac{-8T_\mathrm{h}}{v_n}}+m_\mathrm{v}\phi_n f_n\right]\sin(\omega_n z) \tag{47}$$

Consolidation of the smeared soil

Again by introducing the Fourier sine series, the excess pore-water pressure of the smeared soil can be expressed in fulfilment of equations (6) to (8) of the top and bottom drainage boundary conditions as follows

$$u_\mathrm{s}(r,z,t) = \sum_{n=1}^{\infty} u_{\mathrm{s}n}(r,t)\sin(\omega_n z) \tag{48}$$

$$\bar{u}_\mathrm{s}(z,t) = \sum_{n=1}^{\infty} \bar{u}_{\mathrm{s}n}(t)\sin(\omega_n z) \tag{49}$$

where $u_{\mathrm{s}n}$ and $\bar{u}_{\mathrm{s}n}$ are their corresponding Fourier coefficients.

By using the method of separation of variables, the following equation can be written

$$u_{sn}(r,t) = A_{sn}(r)B_{sn}(t) \tag{50}$$

Following the same derivation procedures as above for the consolidation of the undisturbed soil, the following solution to equation (50) for the consolidation of the smeared soil can be obtained

$$A_{sn}(r) = \lambda_{sn}\phi_{sn}[c_{3n}I_0(\mu_{sn}r) + c_{4n}K_0(\mu_{sn}r) + 1] \tag{51}$$

$$B_{sn}(t) = \frac{1}{\lambda_{sn}\phi_{sn}}\left[a_{sn}\mathrm{e}^{\frac{-8T_{\mathrm{sh}}}{v_{sn}}} + m_{\mathrm{sv}}\phi_{sn}f_n\right] \tag{52}$$

where λ_{sn} is the separation constant; c_{3n}, c_{4n} and a_{sn} are the constants of integration to be determined; and

$$\phi_{sn} = \frac{\gamma_{\mathrm{w}}}{k_{\mathrm{sv}}\omega_n^2} \tag{53}$$

$$\mu_{sn}^2 = \frac{k_{\mathrm{sv}}\omega_n^2}{k_{\mathrm{sh}}} \tag{54}$$

$$T_{\mathrm{sh}} = \frac{k_{\mathrm{sh}}t}{m_{\mathrm{sv}}\gamma_{\mathrm{w}}(2r_{\mathrm{s}})^2} \tag{55}$$

$$v_{sn} = \frac{2\Omega_{sn}}{(\mu_{sn}r_{\mathrm{s}})^2} \tag{56}$$

$$\Omega_{sn} = 1 + \frac{\begin{array}{c}2c_{3n}[\mu_{sn}r_{\mathrm{s}}I_1(\mu_{sn}r_{\mathrm{s}}) - \mu_{sn}r_{\mathrm{d}}I_1(\mu_{sn}r_{\mathrm{d}})]\\ -2c_{4n}[\mu_{sn}r_{\mathrm{s}}K_1(\mu_{sn}r_{\mathrm{s}}) - \mu_{sn}r_{\mathrm{d}}K_1(\mu_{sn}r_{\mathrm{d}})]\end{array}}{(\mu_{sn}r_{\mathrm{s}})^2 - (\mu_{sn}r_{\mathrm{d}})^2} \tag{57}$$

Thus, equation (48) can be rewritten as

$$\begin{aligned} u_{\mathrm{s}}(r,z,t) = \sum_{n=1}^{\infty} & [c_{3n}I_0(\mu_{sn}r) + c_{4n}K_0(\mu_{sn}r) + 1] \\ & \times \left[a_{sn}\mathrm{e}^{\frac{-8T_{\mathrm{sh}}}{v_{sn}}} + m_{\mathrm{sv}}\phi_{sn}f_n\right]\sin(\omega_n z) \end{aligned} \tag{58}$$

In the following sections, the constants of integration in equations (47) and (58) are determined according to the initial and boundary conditions, together with the equations of drain resistance and interface drainage.

Initial conditions

Without loss of generality, the initial average excess pore-water pressures for the undisturbed soil and the smeared soil are assumed to be

$$\bar{u}(z,t=0) = \frac{1}{\pi(r_{\mathrm{e}}^2 - r_{\mathrm{s}}^2)}\int_{r_{\mathrm{s}}}^{r_{\mathrm{e}}} u(r,z,t=0)2\pi r\mathrm{d}r = \sigma_0(z) \tag{59}$$

$$\bar{u}_{\mathrm{s}}(z,t=0) = \frac{1}{\pi(r_{\mathrm{s}}^2 - r_{\mathrm{d}}^2)}\int_{r_{\mathrm{d}}}^{r_{\mathrm{s}}} u_{\mathrm{s}}(r,z,t=0)2\pi r\mathrm{d}r = \sigma_0(z) \tag{60}$$

Substituting equation (47) into equation (59) and substituting equation (58) into equation (60) yields

$$a_n = \frac{\sigma_{n,0}}{\Omega_n} - m_{\mathrm{v}}\phi_n\frac{\sigma_{n,1}}{t_{1,1} - t_{1,0}} \tag{61}$$

$$a_{sn} = \frac{\sigma_{n,0}}{\Omega_{sn}} - m_{\mathrm{sv}}\phi_{sn}\frac{\sigma_{n,1}}{t_{1,1} - t_{1,0}} \tag{62}$$

where $\sigma_{n,0}$ is the Fourier coefficient of Fourier series expansions of the initial increase in vertical total stress σ_0 as shown in Fig. 2(a).

In order to ensure continuity of pore-water pressure and flow rate at all times, the time functions for the consolidation of the undisturbed soil and the smeared soil must be the same, that is

$$B_n(t) = B_{sn}(t) \tag{63}$$

Substituting equations (44) and (52) into equation (63) yields

$$\frac{1}{\lambda_n\phi_n}\left(a_n\mathrm{e}^{\frac{-8T_{\mathrm{h}}}{v_n}} + m_{\mathrm{v}}\phi_n f_n\right) = \frac{1}{\lambda_{sn}\phi_{sn}}\left[a_{sn}\mathrm{e}^{\frac{-8T_{\mathrm{sh}}}{v_{sn}}} + m_{\mathrm{sv}}\phi_{sn}f_n\right] \tag{64}$$

Equation (64) requires that

$$\frac{a_n}{a_{sn}} = \frac{\lambda_n\phi_n}{\lambda_{sn}\phi_{sn}} \tag{65}$$

$$\frac{\lambda_n}{\lambda_{sn}} = \frac{m_{\mathrm{v}}}{m_{\mathrm{sv}}} \tag{66}$$

$$\frac{T_{\mathrm{h}}}{v_n} = \frac{T_{\mathrm{sh}}}{v_{sn}} \tag{67}$$

It can be readily proved that by equations (61), (62), (65) and (66), equation (67) is satisfied.

For the initial conditions specified in equation (10) and Fig. 2, that is, $\sigma_0 = 0$ and $\sigma_{n,0} = 0$, equation (61) becomes

$$a_n = -m_{\mathrm{v}}\phi_n\frac{\sigma_{n,1}}{t_{1,1} - t_{1,0}} \tag{68}$$

By substituting equations (33) and (68) into equation (44), the following generalised time function can be derived

$$\begin{aligned} B_n(t) = \frac{m_{\mathrm{v}}\phi_n}{\lambda_n\phi_n}\sum_{i=1}^{L} & \left\{\frac{\sigma_{n,i} - \sigma_{n,i-1}}{t_{i,1} - t_{i,0}}\right. \\ & \left.\times\left[\mathrm{e}^{\frac{-8\left(T_{\mathrm{h}} - T_{\mathrm{h}i,1}\right)}{v_n}H\langle T_{\mathrm{h}} - T_{\mathrm{h}i,1}\rangle} - \mathrm{e}^{\frac{-8\left(T_{\mathrm{h}} - T_{\mathrm{h}i,0}\right)}{v_n}H\langle T_{\mathrm{h}} - T_{\mathrm{h}i,0}\rangle}\right]\right\} \end{aligned} \tag{69}$$

where

$$T_{\mathrm{h}i,j} = \frac{k_{\mathrm{h}}t_{i,j}}{m_{\mathrm{v}}\gamma_{\mathrm{w}}(2r_{\mathrm{e}})^2},\ j = 0, 1 \tag{70}$$

Drain resistance

Substituting equation (58) into equation (3) yields

$$c_{3n} = \frac{1}{\Delta_1 I_1(\mu_{sn}r_{\mathrm{d}}) - I_0(\mu_{sn}r_{\mathrm{d}})} + \Delta_2 c_{4n} \tag{71}$$

where

$$\Delta_1 = \frac{2}{r_{\mathrm{d}}}\frac{k_{\mathrm{sh}}}{k_{\mathrm{d}}}\frac{\mu_{sn}}{\omega_n^2} \tag{72}$$

$$\Delta_2 = \frac{\Delta_1 K_1(\mu_{sn}r_{\mathrm{d}}) + K_0(\mu_{sn}r_{\mathrm{d}})}{\Delta_1 I_1(\mu_{sn}r_{\mathrm{d}}) - I_0(\mu_{sn}r_{\mathrm{d}})} \tag{73}$$

For an ideal drain without drain resistance, $k_{\mathrm{d}} = \infty$, and hence $\Delta_1 = 0$.

Interface continuity

Substituting equations (47) and (58) into equations (4) and (5) and considering equation (64) yield

$$\begin{aligned} & \lambda_n\phi_n[c_{1n}I_0(\mu_n r_{\mathrm{s}}) + c_{2n}K_0(\mu_n r_{\mathrm{s}}) + 1] \\ & \quad = \lambda_{sn}\phi_{sn}[c_{3n}I_0(\mu_{sn}r_{\mathrm{s}}) + c_{4n}K_0(\mu_{sn}r_{\mathrm{s}}) + 1] \end{aligned} \tag{74}$$

$$\begin{aligned} & k_{\mathrm{h}}\lambda_n\phi_n\mu_n[c_{1n}I_1(\mu_n r_{\mathrm{s}}) - c_{2n}K_1(\mu_n r_{\mathrm{s}})] \\ & \quad = k_{\mathrm{sh}}\lambda_{sn}\phi_{sn}\mu_{sn}[c_{3n}I_1(\mu_{sn}r_{\mathrm{s}}) - c_{4n}K_1(\mu_{sn}r_{\mathrm{s}})] \end{aligned} \tag{75}$$

Substituting equation (71) into equations (74) and (75) gives

$$\alpha_n c_{1n} + \beta_n c_{2n} + \Delta_4 = 0 \tag{76}$$

where

$$\alpha_n = I_0(\mu_n r_{\mathrm{s}}) - \sqrt{\frac{k_{\mathrm{h}}k_{\mathrm{v}}}{k_{\mathrm{sh}}k_{\mathrm{sv}}}}I_1(\mu_n r_{\mathrm{s}})\Delta_3 \tag{77}$$

$$\beta_n = K_0(\mu_n r_s) + \sqrt{\frac{k_h k_v}{k_{sh} k_{sv}}} K_1(\mu_n r_s)\Delta_3 \tag{78}$$

$$\Delta_3 = \frac{\Delta_2 I_0(\mu_{sn} r_s) + K_0(\mu_{sn} r_s)}{\Delta_2 I_1(\mu_{sn} r_s) - K_1(\mu_{sn} r_s)} \tag{79}$$

$$\Delta_4 = \frac{m_{sv}}{m_v}\frac{k_v}{k_{sv}}\left\{\frac{\Delta_3 I_1(\mu_{sn} r_s) - I_0(\mu_{sn} r_s)}{\Delta_1 I_1(\mu_{sn} r_d) - I_0(\mu_{sn} r_d)} - 1\right\} + 1 \tag{80}$$

Vertical drainage boundary conditions

Substituting equation (47) into equation (9) yields

$$c_{1n} I_1(\mu_n r_e) - c_{2n} K_1(\mu_n r_e) = 0 \tag{81}$$

The following can be derived from equations (76) and (81)

$$c_{1n} = \frac{\Delta_4 K_1(\mu_n r_e)}{\Delta_n} \tag{82}$$

$$c_{2n} = \frac{\Delta_4 I_1(\mu_n r_e)}{\Delta_n} \tag{83}$$

where

$$\Delta_n = -\alpha_n K_1(\mu_n r_e) - \beta_n I_1(\mu_n r_e) \tag{84}$$

Substituting equations (71), (82) and (83) into equation (74) leads to

$$c_{4n} = \frac{m_v}{m_{sv}}\sqrt{\frac{k_{sv} k_h}{k_v k_{sh}}}\frac{c_{1n} I_1(\mu_n r_s) - c_{2n} K_1(\mu_n r_s)}{\Delta_2 I_1(\mu_{sn} r_s) - K_1(\mu_{sn} r_s)} - \frac{I_1(\mu_{sn} r_s)}{[\Delta_1 I_1(\mu_{sn} r_d) - I_0(\mu_{sn} r_d)][\Delta_2 I_1(\mu_{sn} r_s) - K_1(\mu_{sn} r_s)]} \tag{85}$$

The final solution

Base on equations (30), (34), (36) and (69), equation (15) can be formulated for calculating the excess pore-water pressure of undisturbed soil. Similarly, based on equations (48), (50), (51), (63), (66) and (69), equation (16) can be derived for calculating the excess pore-water pressure of smeared soil.

NOTATION

A, B	functions of radial coordinate and time
a_n	constant of integration
c_{1n}, c_{2n}	constants of integration
D	parameter of drainage boundary
$\mathrm{d}T_h$, $\mathrm{d}T_v$	parameters analogous to the exponential term in Hansbo's radial consolidation equations, and Terzaghi's time factor for vertical consolidation
F_i	time function of loading and unloading
f_n	partial derivative with respect to time of Fourier coefficient of function that describes increase in total stress
H	Heaviside step function
h	depth of vertical drain
I_0, K_0	modified Bessel functions of the first and second kind of order 0
I_1, K_1	modified Bessel functions of the first and second kind of order 1
k_d	hydraulic conductivity of vertical drain
k_h, k_{sh}	horizontal hydraulic conductivity of undisturbed and smeared soil
k_v, k_{sv}	vertical hydraulic conductivity of undisturbed and smeared soil
L	total number of loading ramps
m_v, m_{sv}	coefficient of volume compressibility of undisturbed and smeared soil
R	loading rate
R_L	depth decay rate of the increase in total stress
r, z	radial and vertical coordinates
r_d, r_s, r_e	radii of vertical drain, smear zone and effective influence zone
T_h	time factor
t	elapsed time
$t_{i,0}$, $t_{i,1}$	start and end time of ith loading ramp
$t_{M,1}$, $t_{L,1}$	end time of loading ramp for the maximum load and final load
U_P	degree of dissipation of excess pore-water pressure
U_S	overall average degree of consolidation
u, u_s	excess pore-water pressure of undisturbed and smeared soil
$\bar{u}$, $\bar{u}_s$	average excess pore-water pressure at a given depth
α	undrained strength gain ratio
β_n	temporary variable
γ_w	unit weight of water
Δ_n	temporary variable
Δs_u	gain in undrained strength
$\Delta\sigma'_v$	increase in effective vertical stress
λ_n	separation constant
μ_n	temporary variable
v_n	temporary variable
σ	increase in total stress
$\sigma_{a,i}$, $\sigma_{b,i}$, $\sigma_{c,i}$	coefficients of quadratic depth function of the increase in total stress
σ_i	depth function of the increase in total stress at the end time of ith loading ramp
σ_M, σ_L	maximum and final increase in total stress
ϕ_n	temporary variable
ω_n	parameter designating period of Fourier sine series

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